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THE DYNAMICS OF AN ECOLOGICAL MODEL WITH INFECTIOUS DISEASE

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ABSTRACT

In this article, the dynamical behavior of a three dimensional continuous time eco-epidemiological model is studied. A prey-predator model involving infectious disease in predator population is proposed and analyzed. This model deals with SI infectious disease that transmitted horizontally in predator population. It is assumed that the disease transmitted to susceptible population in two different ways: contact with infected individuals and an external sources. The existence, uniqueness and bounded-ness of the solution of this model are investigated. The local and global stability conditions of all possible equilibrium points are established. The local bifurcation analysis and a Hopf bifurcation around the positive equilibrium point are obtained. Finally, numerical simulations are given to illustrate our obtained analytical results.

Keywords*: Prey-predator model, Eco-epidemiology, Stability, Hopf bifurcation, Simulations*.

I. INTRODUCTION

The effect of disease in ecological system is an important issue from mathematical as well as ecological points of view. So, in recent time ecologists and researchers are paying more and more attention to the development of important tool along with experimental ecology and describe how ecological species are infected. A simple differential equations prey-predator model to describe the population dynamics of two interacting species was first proposed by the Italian mathematician Vito Volterra and the same differential equations were also derived by Alfred Lotka, a chemist. One of the earliest prey-predator models which are based on sound mathematical logic is the Lotka-Volterra model, which forms the basis of many models used in population dynamics. There are four factors in Lotka-Volterra model which are growth rate of prey, predation rate, mortality rate of predator and conversion rate to change prey biomass into predator population as well as prey population, which grows logistically. On the other hand, most models for the transmission of infectious diseases were originated from the classic work of Kermack and Mc Kendrick [1].

Eco-epidemiological modelling provides challenges in both applied mathematics and theoretical ecology. Anderson and May (1986) [2], were the first who merged ecology and epidemiology and formulated a prey-predator model where the prey species were infected by some infectious diseases. The influence of predation on epidemics has not yet been studied considerably, except the works of Anderson and May [3], Hadeler and Freedman [4], Hochberg [5], Venturino [6, 7], Chattopadhyay and Arino [8], Han et al. [9], Xiao and Chen [10], Hethcote et al. [11], Greenhalgh and Haque [12], and Haque and Venturino [13, 14]. Most of these works have dealt with prey-predator models with disease in the prey (except Venturino [6], and Haque and Venturino [13, 14]). Further more, in recent years eco-epidemiological system with disease in predator has become the most interesting part of research among all mathematical models. Such systems governed mainly by continuous time models investigate stability, bounded-ness and persistence. Krishnapada Das et al. and Prasenjit Das et al. [15-16] studied the prey-predator system with disease in the predator population and discussed the chaos in this system. Pierre Auger et al., Pallav et al., and many other authors have studied the prey-predator system with disease in predator [17, 18].

The simplest models contain a bilinear mass action term, quadratic in both the interacting populations, called also Holling type I. This term appears due to the fact that an individual can in principle interact with the whole other population, the product of the two populations is the obvious outcome. We consider the fact that in general a single individual can feed only until the stomach is full, a saturation function indicate the intake of food. The latter can be modeled by using the concept of the ‘‘law of diminishing returns’’ or technically speaking Michaelis-Menten or

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Holling type II term. The present model is a modification of the previous model studied by Cosner et al. [19], allowing a disease to spread among the predator species only.

This paper is organized as follows: In section 2: The mathematical model is given and the bounded-ness of the solution of this model is proved. In section 3: The existence and local stability analysis of all feasible equilibrium points are studied, global stability analysis for the axial and positive equilibrium points by constructing suitable Lyapunov function are presented. In section 4: An application of Sotomayorś theorem [20] for local bifurcation is used to study the occurrence of local bifurcation near the positive equilibrium point. In section 5: The Hopf bifurcation conditions near the positive equilibrium point are derived. In section 6: Numerical simulations are carried out to support our analytical results. Finally, the last section 7, is devoted to the conclusions and discussion.

II. THE MATHEMATICAL MODEL

In this section, a prey-predator system involving an SI epidemic disease in predator population is proposed for study. In the presence of disease, the predator population is divided into two classes: the susceptible individuals $Y(T)$ and the infected individuals $Z(T)$. Here $Y(T)$ represents the density of susceptible predator population at time T while $Z(T)$ represents the density of infected predator population at time T. The prey population, which denotes to their density at time T by $X(T)$, grows logistically with intrinsic growth rate $r > 0$ and environmental carrying capacity $k > 0$. In the absence of prey species the predator species (susceptible as well as infected) decay exponentially with a natural death rate $d_1 > 0$. The predator species (susceptible as well as infected) consumes the prey species according to modified Cosner type of functional response with maximum attack rate $a_1 > 0$ and $a_2 > 0$ respectively and half-saturation constant $b > 0$. However it converts the food from prey with a conversion rates $e_1 > 0 \; and \; e_2 > 0$ respectively. The existence of disease may causes death in the infected predator with disease death rate $d_2 > 0$. Further its assumed that the disease transferred horizontally between the predator individuals of the same offspring either by contact between the susceptible individuals and infected individuals with contact infected rate $c_1 > 0$ or through an external sources (food, water, air, others) with external infected rate $c_2 > 0$, this is mean that all the new bourn individuals are susceptible individuals. Finally there is a limited vaccine given to the predator individuals to protect them from incidence by disease with vaccine rate $0 < n < 1$, this is left $(1-n)$ susceptible to the disease. Keeping the above hypothesis in view the dynamics of this system can be describe in the following set of differential equations.

$$\begin{aligned} \frac{dX}{dT} &= rX\left(1-\frac{X}{k}\right) - \frac{a_1XY^2}{b+XY} - \frac{a_2XZ^2}{b+XZ} = F_1(X,Y,Z) \\ \frac{dY}{dT} &= e_1\frac{a_1XY^2}{b+XY} + e_2\frac{a_2XZ^2}{b+XZ} - (1-n)Y[c_1Z+c_2] - d_1Y = F_2(X,Y,Z) \\ \frac{dZ}{dT} &= (1-n)Y[c_1Z+c_2] - (d_1+d_2)Z = F_3(X,Y,Z) \end{aligned} \qquad (1)$$

here $X(0) > 0, Y(0) > 0 \; and \; Z(0) > 0$. The flow of disease in system (1) can be described in the following block diagram.

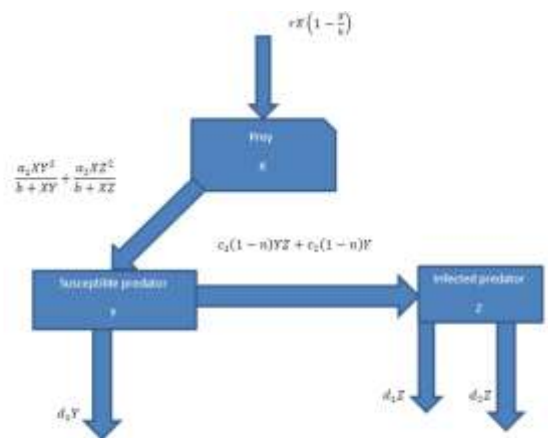

Fig. (1): Block diagram for prey-predator model given by system (1).

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Clearly, system (1) included (12) parameters, which make the analysis difficult. So, in order to simplify the system, the number of parameters is reduced to (10) by using the following dimensionless variables.

$$t = rT, x = \frac{X}{k}, y = \frac{Y}{k}, z = \frac{Z}{k}$$

Thus we obtain:

$$\begin{aligned} \frac{dx}{dt} &= x\left[1 - x - \frac{w_1 y^2}{w_3 + xy} - \frac{w_2 z^2}{w_3 + xz}\right] = x f_1(x,y,z) \\ \frac{dy}{dt} &= \frac{e_1 w_1 x y^2}{w_3 + xy} + \frac{e_2 w_2 x z^2}{w_3 + xz} - w_4 y - (1-n) y [w_5 z + w_6] = f_2(x,y,z) \\ \frac{dz}{dt} &= (1-n) y [w_5 z + w_6] - [w_4 + w_7] z = f_3(x,y,z) \end{aligned} \qquad (2)$$

here $w_1 = \frac{a_1}{r}, w_2 = \frac{a_2}{r}, w_3 = \frac{b}{k^2}, w_4 = \frac{d_1}{r}, w_5 = \frac{c_1 k}{r}, w_6 = \frac{c_2}{r}$, and $w_7 = \frac{d_2}{r}$ represent the dimensionless parameters of the system (2). Moreover the initial condition of system (2) may be taken as any point in the region R_+^3 . The interaction functions in the right hand side of system (2) are continuous and have continuous partial derivatives on R_+^3, Therefore these functions are Lipschitizian on R_+^3, and hence the solution of the system (2) exists and is unique. Further, in the following theorem, the bounded-ness of all the solutions of the system (2) in R_+^3 is established.

Theorem 1. All the solutions of the system (2), which initiate in R_+^3 are uniformly bounded.

Proof. Let $(x(t), y(t), z(t))$ be any solution of the system (2) with non- negative initial condition (x_0, y_0, z_0). According to the first equation of system (2) we have

$$\frac{dx}{dt} \le x(1-x)$$

Then according to the theory of differential inequality [21], we have $Sup\, x(t) \le \mathrm{M}, \forall t \ge 0$, here $M = \max\{x_0, 1\}$. Define the function $G(t) = x(t) + y(t) + z(t)$, then the time derivative of $G(t)$ along the solution of system (2) is $\frac{dG}{dt} \le 2 - \mu G$ where $\mu = \min\{1, w_4\}$ and this gives that $\frac{dG}{dt} + \mu G \le 2$. Again, due to the theory of differential inequality we obtain

$$G(t) \le \frac{2}{\mu} + (G_0 - \frac{2}{\mu}) e^{-\mu t}$$

where $G_0 = (x(0), y(0), z(0))$. Thus, $\forall t \ge 0$ we have that $0 \le G(t) \le \frac{2}{\mu}$ .Hence all solutions of system (2) are uniformly bounded and therefore we have finished the proof.

III. THE STABILITY ANALYSIS OF SYSTEM

In thissection the existence and stability analysis of allfeasible equilibrium points of system (2) are studied. Its observed that system (2) has at most three equilibrium points, which can be stated as follows

1. The trivial equilibrium point, which is denoted by $E_0 = (0,0,0)$ always exists.
2. The axial equilibrium point, which is denoted by$E_1 = (1,0,0)$ always exists.
3. The positive equilibrium point, say $E_2 = (x_2^*, y_2^*, z_2^*)$, of the system (2) exists if there is a positive solution that denoted by (x_2^*, y_2^*, z_2^*) to the following set of equations

$$1 - x - \frac{w_1 y^2}{w_3 + xy} - \frac{w_2 z^2}{w_3 + xz} = 0 \qquad (3a)$$

$$\frac{e_1 w_1 x y^2}{w_3 + xy} + \frac{e_2 w_2 x z^2}{w_3 + xz} - w_4 y - (1-n) y [w_5 z + w_6] = 0 \qquad (3b)$$

$$(1-n) y [w_5 z + w_6] - [w_4 + w_7] z = 0 \qquad (3c)$$

From equation (3c) we get

$$y = \frac{[w_4 + w_7] z}{(1-n)[w_5 z + w_6]} \qquad (4)$$

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Now, by substituting equation (4) in equations (3a) and (3b) yield the following two isoclines

$$f(x,z) = A_1z^4 + A_2z^3 + A_3z^2 + A_4z + A_5xz + A_6x^2 z + A_7xz^2 + A_8xz^3 + A_9xz^4 + A_{10}x^2z^2 + A_{11}x^2z^3 + A_{12}x^3z^2 + A_{13}x^3z^3 + A_{14}x + A_{15} = 0 \tag{5a}$$

$$g(x,z) = B_1z^3 + B_2z^2 + B_3z + B_4xz^2 + B_5xz^3 + B_6xz^4 + B_7x^2z^3 + B_8x^2z^4 = 0 \tag{5b}$$

Where $A_1 = -w_2(1-n)^2{w_5}^2w_3$

$A_2 = -2w_2(1-n)^2w_5w_3w_6$

$A_3 = {w_3}^2{w_5}^2(1-n)^2 - w_1[w_4+w_7]^2w_3 - w_2(1-n)^2{w_6}^2w_3$

$A_4 = 2{w_3}^2w_5(1-n)^2w_6$

$A_5 = (1-n)w_3w_6[w_4+w_7] + (1-n)^2{w_6}^2w_3 -2(1-n)^2{w_3}^2w_5w_6$

$A_6 = -(1-n)w_3w_6[w_4+w_7] - (1-n)^2{w_6}^2w_3$

$A_7 = (1-n)w_3w_5[w_4+w_7] + 2(1-n)^2w_5w_3w_6 -(1-n)^2{w_3}^2{w_5}^2$

$A_8 = (1-n)^2{w_5}^2w_3 - w_1[w_4+w_7]^2 - w_2(1-n)w_6[w_4+w_7]$

$A_9 = -w_2(1-n)w_5[w_4+w_7]$

$A_{10} = (1-n)w_6[w_4+w_7] - (1-n)w_3w_5[w_4+w_7] -2(1-n)^2w_5w_3w_6$

$A_{11} = (1-n)w_5[w_4+w_7] - (1-n)^2{w_5}^2w_3$

$A_{12} = -(1-n)w_6[w_4+w_7]$

$A_{13} = -(1-n)w_5[w_4+w_7]$

$A_{14} = -(1-n)^2{w_3}^2{w_6}^2$

$A_{15} = {w_3}^2{w_6}^2(1-n)^2$

and $B_1 = -[w_4+w_7](1-n)^2{w_3}^2{w_5}^2$

$B_2 = -w_4[w_4+w_7]{w_3}^2(1-n)w_5 - 2[w_4+w_7](1-n)^2{w_3}^2w_5w_6$

$B_3 = -w_4[w_4+w_7]{w_3}^2(1-n)w_6 - [w_4+w_7](1-n)^2{w_3}^2{w_6}^2$

$B_4 = e_1w_1[w_4+w_7]^2w_3 + e_2w_2(1-n)^2{w_6}^2w_3 -w_4[w_4+w_7]^2w_3 - w_4[w_4+w_7]w_3(1-n)w_6 -[w_4+w_7]^2(1-n)w_3w_6 - [w_4+w_7](1-n)^2{w_6}^2w_3$

$B_5 = 2e_2w_2(1-n)^2w_5w_3w_6 - w_4[w_4+w_7]w_3(1-n)w_5 - [w_4+w_7]^2(1-n)w_3w_5 - 2[w_4+w_7](1-n)^2w_5w_3w_6$

$B_6 = e_2w_2(1-n)^2{w_5}^2w_3 - [w_4+w_7](1-n)^2{w_5}^2w_3$

$B_7 = e_1w_1[w_4+w_7]^2 + e_2w_2(1-n)w_6[w_4+w_7] -w_4[w_4+w_7]^2 - [w_4+w_7]^2(1-n)w_6$

$B_8 = e_2w_2(1-n)w_5[w_4+w_7] - [w_4+w_7]^2(1-n)w_5$

From equation (5a) we note that as $z \to 0$, then $x \to x_1$, with

$$x_1 = -\frac{A_{15}}{A_{14}} > 0 \tag{6}$$

Also from the equation (5a), we have $\frac{dx}{dz} = -\frac{\partial f}{\partial z}\Big/\frac{\partial f}{\partial x}$ is negative provided that

$$\frac{\partial f}{\partial z} > 0 \text{ and } \frac{\partial f}{\partial x} > 0 \quad \text{or} \quad \frac{\partial f}{\partial z} < 0 \text{ and } \frac{\partial f}{\partial x} < 0 \tag{7}$$

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Again from equation (5b), we note that when $z \to 0$, then $x \to 0$. Moreover we have $\frac{dx}{dz} = -\frac{\partial g}{\partial z} / \frac{\partial g}{\partial x}$ is positive provided that

$$\frac{\partial g}{\partial z} > 0 \text{ and } \frac{\partial g}{\partial x} < 0 \quad \text{or} \quad \frac{\partial g}{\partial z} < 0 \text{ and } \frac{\partial g}{\partial x} > 0 \qquad (8)$$

From the above analysis we note that two isoclines (5a) and (5b) intersect at a unique positive point denoted by (x_2^*, z_2^*) provided that conditions (7)-(8) are satisfied. Knowing the values of x_2^*andz_2^*, the value of y_2^* can be calculated from equation (4) above, this is presents the conditions of existence of $E_2 = (x_2^*, y_2^*, z_2^*)$.

Now the local stability analysis of the above feasible equilibrium points of system (2) is studied analytically with the help of Linearization method. Note that it is easy to verify that, the Jacobian matrix of system (2) at the trivial equilibrium point $E_0 = (0,0,0)$ can be written in the form:

$$J_0 \equiv J(E_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -w_4 - (1-n)w_6 & 0 \\ 0 & (1-n)w_6 & -(w_4 + w_7) \end{pmatrix} \qquad (9)$$

Clearly,J_0 has two negative eigenvalues $\lambda_y = -w_4 - (1-n)w_6$, $\lambda_z = -[w_4 + w_7]$ and one positive eigenvalue in the x-direction $(\lambda_x = 1)$, so the equilibrium point E_0 is unstable saddle point.

The Jacobian matrix of system (2) at the axial equilibrium point $E_1 = (1,0,0)$ can be written as:

$$J_1 \equiv J(E_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -w_4 - (1-n)w_6 & 0 \\ 0 & (1-n)w_6 & -(w_4 + w_7) \end{pmatrix} \qquad (10)$$

Clearly, J_1 has three negative eigenvalues $\lambda_{1x} = -1$, $\lambda_{1y} = -w_4 - (1-n)w_6$ and $\lambda_{1z} = -[w_4 + w_7]$, so the equilibrium point E_1 is always locally asymptotically stable.
The Jacobian matrix of system (2) at positive equilibrium point E_2can be written as

$$J_2 \equiv J(E_2) = (a_{ij})_{3\times3} \qquad (11a)$$

here
$$a_{11} = x_2^*\left[-1 + \frac{w_1 {y_2^*}^3}{{M_1^*}^2} + \frac{w_2 {z_2^*}^3}{{M_2^*}^2}\right]$$
$$a_{12} = x_2^*\left[-\frac{2w_1w_3y_2^* + w_1x_2^*{y_2^*}^2}{{M_1^*}^2}\right] < 0$$
$$a_{13} = x_2^*\left[-\frac{2w_2w_3z_2^* + w_2x_2^*{z_2^*}^2}{{M_2^*}^2}\right] < 0$$
$$a_{21} = \frac{e_1w_1w_3{y_2^*}^2}{{M_1^*}^2} + \frac{e_2w_2w_3{z_2^*}^2}{{M_2^*}^2} > 0$$
$$a_{22} = \frac{2e_1w_1w_3x_2^*y_2^* + e_1w_1{x_2^*}^2{y_2^*}^2}{{M_1^*}^2} - w_4 - (1-n)[w_5z_2^* + w_6]$$
$$a_{23} = \frac{2e_2w_2w_3x_2^*z_2^* + e_2w_2{x_2^*}^2{z_2^*}^2}{{M_2^*}^2} - (1-n)w_5y_2^*$$
$$a_{31} = 0$$
$$a_{32} = (1-n)[w_5z_2^* + w_6] > 0$$
$$a_{33} = (1-n)y_2^*w_5 - [w_4 + w_7]$$
$$M_1^* = w_3 + x_2^*y_2^* and \ M_2^* = w_3 + x_2^*z_2^*$$

Then the characteristic equation of J_2 can be written as:

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0 \quad (11b)$$

Where $C_1 = -(a_{11} + a_{22} + a_{33})$,
$C_2 = a_{22}a_{33} - a_{23}a_{32} + a_{11}a_{33} + a_{11}a_{22} - a_{12}a_{21}$
$C_3 = a_{33}(a_{12}a_{21} - a_{11}a_{22}) + a_{32}(a_{11}a_{23} - a_{13}a_{21})$

However

$$\Delta = C_1C_2 - C_3 = -2a_{11}a_{22}a_{33} - a_{22}{}^2(a_{33} + a_{11}) - (a_{22} + a_{11})(a_{33}{}^2 - a_{12}a_{21}) - a_{22}a_{11}{}^2 + a_{22}a_{23}a_{32} - a_{33}a_{11}{}^2 + a_{32}(a_{33}a_{23} + a_{13}a_{21})$$

Now according to Routh-Hurwitz criterion the equilibrium point E_2 is locally asymptotically stable, provided that $C_1 > 0, C_3 > 0$ and $\Delta = C_1C_2 - C_3 > 0$. Hence straightforward computation shows that the equilibrium point E_2 is locally asymptotically stable provided that

$$\frac{w_1 y_2^{*3}}{M_1{}^{*2}} + \frac{w_2 z_2^{*3}}{M_2{}^{*2}} < 1 \quad (12a)$$

$$\frac{2e_1w_1w_3x_2^*y_2^* + e_1w_1x_2^{*2}y_2^{*2}}{M_1{}^{*2}} < w_4 \quad (12b)$$

$$\frac{2e_2w_2w_3x_2{}^*z_2{}^* + e_2w_2x_2{}^{*2}z_2{}^{*2}}{M_2{}^{*2}} < (1-n)y_2^*w_5 < w_4 + w_7 \quad (12c)$$

$$\left((1-n)y_2^*w_5 - [w_4 + w_7]\right)\left(\frac{e_2w_2x_2^*z_2^*(2w_3 + x_2^*z_2^*)}{M_2{}^{*2}} - (1-n)y_2^*w_5\right) > x_2^*w_2z_2^*\frac{(2w_3 + x_2^*z_2^*)}{M_2{}^{*2}}\left(\frac{e_1w_1w_3y_2^{*2}}{M_1{}^{*2}} + \frac{e_2w_2w_3z_2^{*2}}{M_2{}^{*2}}\right) \quad (12d)$$

Clearly, the conditions (12a)-(12c) guarantee that a_{11}, a_{22}, a_{33} and a_{23} are negative and hence $C_1 > 0$ and $C_3 > 0$. However, the conditions (12a)-(12d) guarantee that $C_1C_2 - C_3 > 0$.

Now the global stability analysis of the feasible equilibrium points of system (2) is studied analytically with the help of Lyapunov method as shown in the following theorems.

Theorem 2. Assume that the axial equilibrium point E_1 is a locally asymptotically stable in R_+^3, then E_1 is a globally asymptotically stable on the sub region of R_+^3 that satisfy the following conditions:

$$e_1 \geq e_2 \quad (13a)$$

$$y < \frac{w_3w_4}{e_1w_1} \quad (13b)$$

$$z < \frac{w_3[w_4 + w_7]}{e_1w_2} \quad (13c)$$

Proof. Consider the following positive definite function:

$$L_1 = b_1[x - 1 - \ln x] + b_2y + b_3z$$

where $b_i; i = 1,2,3$ are positive constants to be determined. Clearly, $L_1: R_+^3 \to R$ is continuously differentiable function so that $L_1(1, 0, 0) = 0$ and $L_1(x, y, z) > 0$ for all $(x, y, z) \in R_+^3$ with $(x, y, z) \neq (1, 0, 0)$.
Therefore by differentiating this function with respect to the time, we get:

$$\frac{dL_1}{dt} = b_1\frac{(x-1)}{x}\frac{dx}{dt} + b_2\frac{dy}{dt} + b_3\frac{dz}{dt}$$

Substituting the value of $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ from system (2) in the above equation and after doing some algebraic manipulation, we get that

$$\frac{dL_1}{dt} = -b_1(x-1)^2 - \frac{w_1xy^2}{w_3+xy}(b_1 - b_2e_1) + b_1\frac{w_1y^2}{w_3+xy}$$
$$-\frac{w_2xz^2}{w_3+xz}(b_1 - b_2e_2) + b_1\frac{w_2z^2}{w_3+xz} - b_2w_4y$$
$$-(1-n)y[w_5z + w_6](b_2 - b_3) - b_3[w_4 + w_7]z$$

So by choosing the positive constants as below:

$$b_1 = e_1; \quad b_2 = b_3 = 1$$

and using condition (13a) we obtain that:

$$\frac{dL_1}{dt} \le -e_1(x-1)^2 - y\left[w_4 - e_1\frac{w_1y}{w_3}\right] - z\left[[w_4 + w_7] - e_1\frac{w_2z}{w_3}\right]$$

Now, it is easy to verify that, condition (13b) guarantees the negativity of the second term while condition (13c) guarantees the negativity of the third term.
So; $\frac{dL_1}{dt}$ is negative definite and hence $L_1$is a Lyapunov function. Thus E_1 is a globally asymptotically stable on the sub region of R_+^3 that satisfy the given conditions.

Theorem 3. Assume that the positive equilibrium point E_2 is a locally asymptotically stable in R_+^3 thenE_2 is a globally asymptotically stable on the sub region of R_+^3 that satisfy the following conditions:

$$P_{11} > 0 \tag{14a}$$
$$P_{22} > 0 \tag{14b}$$
$$P_{33} > 0 \tag{14c}$$
$$P_{12}{}^2 < P_{11}P_{22} \tag{14d}$$
$$P_{13}{}^2 < P_{11}P_{33} \tag{14e}$$
$$P_{23}{}^2 < P_{22}P_{33} \tag{14f}$$

where $P_{11} = 1 - \frac{w_1{y_2^*}^2y}{M_1M_1^*} - \frac{w_2{z_2^*}^2z}{M_2M_2^*}$

$$P_{12} = \frac{e_2w_2w_3{z_2^*}^2}{M_2M_2^*} + \frac{e_1w_1w_3{y_2^*}^2 - w_1w_3(y+y_2^*) - w_1y_2^*yx_2^*}{M_1M_1^*}$$
$$P_{13} = \frac{w_2w_3(z+z_2^*) + w_2z_2^*zx_2^*}{M_2M_2^*}$$
$$P_{22} = (1-n)(z_2^*w_5 + w_6) + w_4 - \frac{(e_1w_1x_2^*y_2^*xy + e_1w_1w_3x(y+y_2^*))}{M_1M_1^*}$$
$$P_{23} = -(1-\mathrm{n})w_5y + (1-n)(z_2^*w_5 + w_6) + \frac{e_2w_2w_3x(z+z_2^*) + e_2w_2x_2^*z_2^*xz}{M_2M_2^*}$$
$$P_{33} = w_4 + w_7 - (1-n)w_5y$$
$$M_1 = w_3 + xy, \quad M_1^* = w_3 + x_2^*y_2^*$$
$$M_2 = w_3 + xz, \quad M_2^* = w_3 + x_2^*z_2^*$$

Proof. Consider the following positive definite function:

$$L_2 = \left(x - x_2^* - x_2^*\ln\frac{x}{x_2^*}\right) + \frac{(y-y_2^*)^2}{2} + \frac{(z-z_2^*)^2}{2}$$

Clearly, $L_2: R_+^3 \to R$ is continuously differentiable function so that $L_2(x_2^*, y_2^*, z_2^*) = 0$ and $L_2(x,y,z) > 0$ for all $(x, y, z) \in R_+^3$ with $(x, y, z) \ne (x_2^*, y_2^*, z_2^*)$.

Therefore by differentiating this function with respect to the time, we get:

$$\frac{dL_2}{dt} = \frac{(x-x_2^*)}{x}\frac{dx}{dt} + (y-y_2^*)\frac{dy}{dt} + (z-z_2^*)\frac{dz}{dt}$$

Substituting the value of $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ from system (2) in the above equation and after doing some algebraic manipulation, we get that

$$\frac{dL_2}{dt} = -P_{11}(x - x_2^*)^2 + P_{12}(x - x_2^*)(y - y_2^*) - P_{22}(y - y_2^*)^2$$
$$-P_{13}(x - x_2^*)(z - z_2^*) + P_{23}(y - y_2^*)(z - z_2^*) - P_{33}(z - z_2^*)^2$$

Now, it is easy to verify that, under the given conditions we can rewrite the above equation as a sum of quadratics. Then

$$\frac{dL_2}{dt} \le -\left[\sqrt{\frac{1}{2}P_{11}}(x - x_2^*) - \sqrt{\frac{1}{2}P_{22}}(y - y_2^*)\right]^2$$
$$-\left[\sqrt{\frac{1}{2}P_{11}}(x - x_2^*) + \sqrt{\frac{1}{2}P_{33}}(z - z_2^*)\right]^2$$
$$-\left[\sqrt{\frac{1}{2}P_{22}}(y - y_2^*) - \sqrt{\frac{1}{2}P_{33}}(z - z_2^*)\right]^2$$

So; $\frac{dL_2}{dt}$ is negative definite and hence L_2 is a Lyapunov function. ThusE_2 is a globally asymptotically stable on the sub region of R_+^3 that satisfy the given conditions.

IV. THE LOCAL BIFURCATION ANALYSIS

In this section, the effect of varying the parameters values on the dynamical behavior of the system (2) around each equilibrium points is studied analytically. It is well known that the existence of non-hyperbolic equilibrium point of system (2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorem an application to the Sotomayorś theorem [20] for local bifurcation is applied with the parameter value that transfers the equilibrium point from hyperbolic to non-hyperbolic. Before we go to study the local bifurcation that may occurs in system (2) near the non-hyperbolic equilibrium point the following calculations are needed.

It is easy to check that for any non-zero vector $V = (v_1, v_2, v_3)^T$ we have:

$$D^2 f(V,V) = \left(d_{ij}\right)_{3\times 1} \tag{15}$$

here $f = (f_1, f_2, f_3)^T$ with $f_i; i = 1,2,3$ is given in system (2).

$$d_{11} = {v_1}^2 K_1 - K_2 v_1 v_2 - K_3 {v_2}^2 - K_4 v_1 v_3 - K_5 {v_3}^2$$
$$d_{21} = {v_1}^2 K_6 + K_2 e_1 v_1 v_2 + K_3 e_1 {v_2}^2 + K_4 e_2 v_1 v_3 + K_5 e_2 {v_3}^2 - K_7 v_2 v_3$$
$$d_{31} = K_7 v_2 v_3$$

and $K_1 = \frac{2w_1 w_3 y^3}{{M_1}^3} + \frac{2w_2 w_3 z^3}{{M_2}^3} - 2$; $K_2 = \frac{4w_1 {w_3}^2 y}{{M_1}^3}$; $K_3 = \frac{2w_1 {w_3}^2 x}{{M_1}^3}$; $K_4 = \frac{4w_2 {w_3}^2 z}{{M_2}^3}$

$K_5 = \frac{2w_2 {w_3}^2 x}{{M_2}^3}$; $K_6 = -\frac{2e_1 w_1 w_3 y^3}{{M_1}^3} - \frac{2e_2 w_2 w_3 z^3}{{M_2}^3}$; $K_7 = 2(1-n)w_5$

While

$$D^3 f(V,V,V) = \left(e_{ij}\right)_{3\times 1} \tag{16}$$

where $e_{11} = {v_1}^3 N_1 + {v_1}^2 v_2 N_2 - v_1 {v_2}^2 N_3 + {v_1}^2 v_3 N_4 - v_1 {v_3}^2 N_5 + N_6 {v_2}^3 + N_7 {v_3}^3$

$$e_{21} = {v_1}^3 N_8 - {v_1}^2 v_2 e_1 N_2 + v_1 {v_2}^2 e_1 N_3 - {v_1}^2 v_3 e_2 N_4$$
$$+ v_1 {v_3}^2 e_2 N_5 - e_1 N_6 {v_2}^3 - e_2 N_7 {v_3}^3$$
$$e_{31} = 0$$

with $N_1 = -\frac{6w_1 w_3 y^4}{{M_1}^4} - \frac{6w_2 w_3 z^4}{{M_2}^4}$; $N_2 = \frac{18 w_1 {w_3}^2 y^2}{{M_1}^4}$; $N_3 = \frac{6w_1 {w_3}^2 (w_3 - 2xy)}{{M_1}^4}$

$N_4 = \frac{18w_2{w_3}^2z^2}{{M_2}^4}$; $N_5 = \frac{6w_2{w_3}^2(w_3-2xz)}{{M_2}^4}$; $N_6 = \frac{6w_1{w_3}^2x^2}{{M_1}^4}$; $N_7 = \frac{6w_2{w_3}^2x^2}{{M_2}^4}$

$$N_8 = \frac{6e_1w_1w_3y^4}{{M_1}^4} + \frac{6e_2w_2w_3z^4}{{M_2}^4}$$

Note that, according to the Jacobian matrices of system (2) at E_0 and E_1, those given by Eq. (9) and (10) respectively, it is clear that J_0 and J_1 have always three non-zero eigenvalues. So, the equilibrium points E_0 and E_1 cannot be non-hyperbolic equilibrium points. Thus, system (2) is structurally stable at E_0 and E_1 and hence it has no bifurcation at them. Accordingly in the following theorem we will study the occurrence of local bifurcation around the positive equilibrium point.

Theorem 4. Suppose that the conditions (12a) and (12b) along with the following conditions are satisfied:

$$(1-n){y_2}^*w_5 > \max\left\{\frac{2e_2w_2w_3\, {x_2}^*{z_2}^*+e_2w_2{x_2}^{*2}{z_2}^{*2}}{M_2^{*2}}, w_4 - \frac{a_{32}S_2}{S_1}\right\} \quad (17a)$$

$$\beta_3{\beta_1}^2{K_1}^* + \beta_4{\beta_1}^2{K_6}^* + ({K_2}^*\beta_1 + {K_3}^*\beta_2)\beta_2(\beta_4e_1 - \beta_3) + ({K_4}^*\beta_1 + {K_5}^*)(\beta_4e_2 - \beta_3) + {K_7}^*\beta_2(1-\beta_4) \neq 0 \quad (17b)$$

here $S_1 = a_{12}a_{21} - a_{11}a_{22}$; $S_2 = a_{11}a_{23} - a_{13}a_{21}$ while $\beta_i; i = 1,2,3,4$ are given in the proof. Then system (2) undergoes saddle-node bifurcation around E_2, but neither transcritical bifurcation nor pitchfork bifurcation can occur, as the parameter w_7 passes through the value ${w_7}^* = a_{32}S_2\left[-\frac{w_4}{a_{32}S_2} + \frac{(1-n){y_2}^*w_5}{a_{32}S_2} + \frac{1}{S_1}\right]$.

Proof. Note that the characteristic equation given by Eq. (11b) having zero root $(say\ \lambda = 0)$ if and only if $C_3 = 0$ and then E_2 becomes a non-hyperbolic equilibrium point. Now the Jacobian matrix of system (2) at the equilibrium point E_2 with parameter $w_7 = {w_7}^*$ becomes

$${J_2}^* = J_2(E_2, {w_7}^*) = \left(a_{ij}\right)_{3\times 3} \text{ with } a_{33} = -\frac{a_{32}S_2}{S_1}$$

where a_{ij} for all $i, j = 1, 2, 3$ is given by Eq. (11a).
Note that, the conditions (12a)-(12b) and (17a) guarantee that a_{11}, a_{22}, a_{23} and S_1 are negative, while ${w_7}^*$ and S_2 are positive. Consequently, a_{33} becomes positive with $C_3 = 0$. Therefore ${J_2}^*$ has a zero eigenvalue, say $\lambda = 0$, as the parameter w_7 passes through the value ${w_7}^*$.
Let $V^* = ({v_1}^*, {v_2}^*, {v_3}^*)^T$ be the eigenvector corresponding to the eigenvalue $\lambda = 0$. Thus $({J_2}^* - \lambda I)V^* = 0$, which gives:

$${v_1}^* = \beta_1{v_3}^*, {v_2}^* = \beta_2{v_3}^* \text{ with } \beta_1 = -\frac{(a_{12}S_2 + S_1a_{13})}{a_{11}S_1} \in \mathcal{R} \text{ and } \beta_2 = \frac{S_2}{S_1} < 0$$

while ${v_3}^*$ any non-zero real number.

Let $\Psi^* = ({\Psi_1}^*, {\Psi_2}^*, {\Psi_3}^*)^T$ be the eigenvector associated with the eigenvalue $\lambda = 0$ of the matrix ${J_2}^{*T}$. Then we have $\left({J_2}^{*T} - \lambda I\right)\Psi^* = 0$, which gives :

$${\Psi_1}^* = \beta_3{\Psi_3}^*, {\Psi_2}^* = \beta_4{\Psi_3}^* \text{ with } \beta_3 = -\frac{a_{21}a_{32}}{S_1} > 0 \text{ and } \beta_4 = \frac{a_{32}a_{11}}{S_1} > 0$$

while ${\Psi_3}^*$ any non-zero real number.

Now, consider

$$\frac{\partial f}{\partial w_7} = f_{w_7}(X, w_7) = \left(\frac{\partial f_1}{\partial w_7}, \frac{\partial f_2}{\partial w_7}, \frac{\partial f_3}{\partial w_7}\right)^T = (0, 0, -z)^T$$

So, $f_{w_7}(E_2, {w_7}^*) = (0, 0, -{z_2}^*)^T$ and hence

$$(\Psi^*)^T f_{w_7}(E_2, {w_7}^*) = -{\Psi_3}^*{z_2}^* \neq 0$$

So, according to Sotomayorś theorem the transcritical bifurcation and pitchfork bifurcation cannot occur. While the first condition of the saddle-node bifurcation is satisfied.

Now, by substituting V^* in Eq. (15) we get

$$D^2 f(E_2, w_7^*)(V^*, V^*) = \left(d_{ij}^*\right)_{3\times 1}$$

Where $d_{11}^* = \left[\beta_1{}^2 K_1{}^* - K_2{}^*\beta_1\beta_2 - K_3{}^*\beta_2{}^2 - K_4{}^*\beta_1 - K_5{}^*\right](v_3{}^*)^2$;

$$d_{21}^* = \left[\beta_1{}^2 K_6{}^* + K_2{}^* e_1\beta_1\beta_2 + K_3{}^* e_1\beta_2{}^2 + K_4{}^* e_2\beta_1 + K_5{}^* e_2 - K_7{}^*\beta_2\right](v_3{}^*)^2$$
$$d_{31}^* = K_7{}^*\beta_2(v_3{}^*)^2$$

Here $K_i^* = K_i(E_2, w_7^*)$ in Eq. (15) for all $i = 1,2,\ldots,7$. Hence, it is obtain:

$$(\Psi^*)^T[D^2 f(E_2, w_7^*)(V^*, V^*)] = \left[\beta_3\beta_1{}^2 K_1{}^* + \beta_4\beta_1{}^2 K_6{}^* + (K_2{}^*\beta_1 + K_3{}^*\beta_2)\beta_2(\beta_4 e_1 - \beta_3) + (K_4{}^*\beta_1 + K_5{}^*)(\beta_4 e_2 - \beta_3) + K_7{}^*\beta_2(1-\beta_4)\right]\Psi_3{}^*(v_3{}^*)^2$$

So, according to condition (17b) we obtain that $(\Psi^*)^T[D^2 f(E_2, w_7^*)(V^*, V^*)] \neq 0$.
Thus according to Sotomayorś theorem, system (2) undergoes saddle-node bifurcation around E_2 and the proof is complete.

V. THE HOPH BIFURCATION OF ANALYSIS

In this section, the possibility of existence of periodic dynamic in system (2) due to changing the value of one parameter is studied. First, since the Jacobian matrices of system (2) at E_0 *and* E_1, those given by Eq. (9) and Eq. (10) respectively, have always three real eigenvalues then there is no possibility of occurrence of Hopf bifurcation around them. On the other hand the possibility of occurrence of Hopf bifurcation around the positive equilibrium point is discussed in the following theorem.

Theorem 5. Suppose that the conditions (12a), (12b) and (12c) along with the following conditions are satisfied**:**

$$a_{33}a_{23} + a_{13}a_{21} < 0 \qquad (18a)$$
$$(a_{11} + a_{22})(a_{12}a_{21} - a_{11}a_{22}) + a_{22}a_{23}a_{32} + a_{32}a_{13}a_{21} < 0 \qquad (18b)$$

Then system (2) undergoes a Hopf bifurcation around the equilibrium point E_2as the parameter w_7 passes through the value

$$w_7{}^{\sim} = \frac{1}{2I_1}\left(I_2 + \sqrt{(I_2)^2 - 4I_1 I_3}\right) + (1-n)y_2{}^* w_5 - w_4,$$

where $I_i; i = 1,2,3$ are given in the proof.

Proof. Recall that according to the Hopf bifurcation theorem system (2) undergoes a Hopf bifurcation around E_2 provided that the Jacobian matrix J_2, which given in Eq. (11a), have two complex eigenvalues with the third real and negative:

$$\lambda_{1,2}(w_7) = \xi_1(w_7) \pm i\xi_2(w_7);\ \lambda_3 \in \mathcal{R} \text{ and} \lambda_3 < 0 \qquad (19a)$$

Such that:

$$\xi_1(w_7{}^{\sim}) = 0 \qquad (19b)$$
$$\left.\frac{d\xi_1}{dw_7}\right|_{w_7 = w_7{}^{\sim}} \neq 0 \qquad (19c)$$

Now Straightforward computation gives that conditions (12a)-(12c) guarantee that the coefficients of the characteristic equation (11b), C_i for $i = 1,2,3$, are positive. However $\Delta = C_1 C_2 - C_3$ can be rewritten as

$$\Delta = I_1(a_{33})^2 + I_2 a_{33} + I_3$$

where

$$I_1 = -(a_{11} + a_{22}) > 0$$
$$I_2 = -(a_{11} + a_{22})^2 + a_{32}a_{23} < 0$$
$$I_3 = (a_{11} + a_{22})(a_{12}a_{21} - a_{11}a_{22}) + a_{22}a_{23}a_{32} + a_{32}a_{13}a_{21}$$

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Clearly conditions (12a)-(12c) with the conditions (18a) and (18b) guarantee that the equation $\Delta = C_1C_2 - C_3 = 0$ has two opposite sign real roots given by

$$a_{33} = -\frac{I_2}{2I_1} - \frac{\sqrt{(I_2)^2 - 4I_1I_3}}{2I_1} < 0;\ a_{33} = -\frac{I_2}{2I_1} + \frac{\sqrt{(I_2)^2 - 4I_1I_3}}{2I_1} > 0$$

Moreover, since condition (12c) guarantees that a_{33} is negative, hence by using the form of a_{33} in Eq. (11a) we obtain that the equation $\Delta = C_1C_2 - C_3 = 0$ as the parameter w_7 passes through the value $w_7^{\sim}$ that given above.
Accordingly, for $w_7 = w_7^{\sim}$ the characteristic equation (11b) can be written

$$(\lambda + C_1)(\lambda^2 + C_2) = 0 \tag{20}$$

It is easy to verify that Eq. (20) has three roots $\lambda_{1,2} = \pm i\sqrt{C_2}$ and $\lambda_3 = -C_1 < 0$. Therefore condition (19b) holds with $\xi_2(w_7^{\sim}) = \sqrt{C_2} \neq 0$.
Further, for all values of w_7 in the neighborhood of $w_7^{\sim}$, the roots are in general given in form of Eq. (19a) with $\lambda_3(w_7) = -C_1(w_7) < 0$.

Now by substituting the complex eigenvalues given by (19a) in the Eq. (20) and then calculating the derivative with respect to the bifurcation parameter w_7, and then comparing the two sides of the resulting equation and equating their real and imaginary parts, it is obtain that :

$$\begin{aligned} \Psi(w_7)\xi_1'(w_7) - \Phi(w_7)\xi_2'(w_7) &= -\Theta(w_7) \\ \Phi(w_7)\xi_1'(w_7) + \Psi(w_7)\xi_2'(w_7) &= -\Gamma(w_7) \end{aligned} \tag{21}$$

here:

$$\Psi(w_7) = 3(\xi_1(w_7))^2 + 2C_1(w_7)\xi_1(w_7) + C_2(w_7) - 3(\xi_2(w_7))^2$$
$$\Phi(w_7) = 6\xi_1(w_7)\xi_2(w_7) + 2C_1(w_7)\xi_2(w_7)$$
$$\Theta(w_7) = \left(\xi_1(w_7)\right)^2 C_1'(w_7) + C_2'(w_7)\xi_1(w_7) + C_3'(w_7) - C_1'(w_7)\left(\xi_2(w_7)\right)^2$$
$$\Gamma(w_7) = 2\xi_1(w_7)\xi_2(w_7)C_1'(w_7) + C_2'(w_7)\xi_2(w_7)$$

Now by solving the linear system (21) by using Cramerś rule for the unknowns $\xi_1'(w_7)$and$\xi_2'(w_7)$, gives that

$$\xi_1'(w_7) = -\frac{\Theta(w_7)\Psi(w_7) + \Gamma(w_7)\Phi(w_7)}{(\Psi(w_7))^2 + (\Phi(w_7))^2} \tag{22}$$

Therefore the second necessary and sufficient condition of Hopf bifurcation (19c) will be satisfied if and only if

$$\Theta(w_7^{\sim})\Psi(w_7^{\sim}) + \Gamma(w_7^{\sim})\Phi(w_7^{\sim}) \neq 0 \tag{23}$$

Note that for $w_7 = w_7^{\sim}$ we have:

$$\Psi(w_7^{\sim}) = -2C_2(w_7^{\sim})$$
$$\Gamma(w_7^{\sim}) = -(a_{11} + a_{22})\sqrt{C_2(w_7^{\sim})}$$
$$\Phi(w_7^{\sim}) = 2C_1(w_7^{\sim})\sqrt{C_2(w_7^{\sim})}$$
$$\Theta(w_7^{\sim}) = -(a_{11} + a_{22})a_{33} + a_{23}a_{32}$$

Substitution into Eq.(23) gives

$$\begin{aligned} &\Theta(w_7^{\sim})\Psi(w_7^{\sim}) + \Gamma(w_7^{\sim})\Phi(w_7^{\sim}) \\ &\quad = -2C_2(w_7^{\sim})[-(a_{11} + a_{22})a_{33} + a_{23}a_{32}] + 2C_2(w_7^{\sim})[(a_{11} + a_{22})(a_{11} + a_{22} + a_{33})] > 0 \end{aligned}$$

Then the second condition (19c) of the Hopf bifurcation is satisfied. So, system (2) undergoes a Hopf bifurcation around the equilibrium point E_2 at the parameter $w_7 = w_7^{\sim}$ and the proof is complete.

VII. NUMERICAL SIMULATION

In this section, the global dynamics of system (2) is studied numerically for different sets of initial values a long with different sets of parameters values. The objectives of this study are determine the effect of varying the parameters values on the dynamics of system (2) as well as confirm our obtained analytical results. Now for the following hypothetical biologically feasible set of parameters values:

$$\begin{aligned} &w_1 = 1, w_2 = 1, w_3 = 0.4, w_4 = 0.1, \ w_5 = 0.2 \\ &w_6 = 0.1, w_7 = 0.1, e_1 = 0.5, e_2 = 0.5, n = 0.2 \end{aligned} \tag{24}$$

The trajectory of the system (2) is drawn in the Fig. (2a), while the time series for x, y and z starting from three different initial points are drawn in the Fig. (2b)-Fig. (2d) respectively.

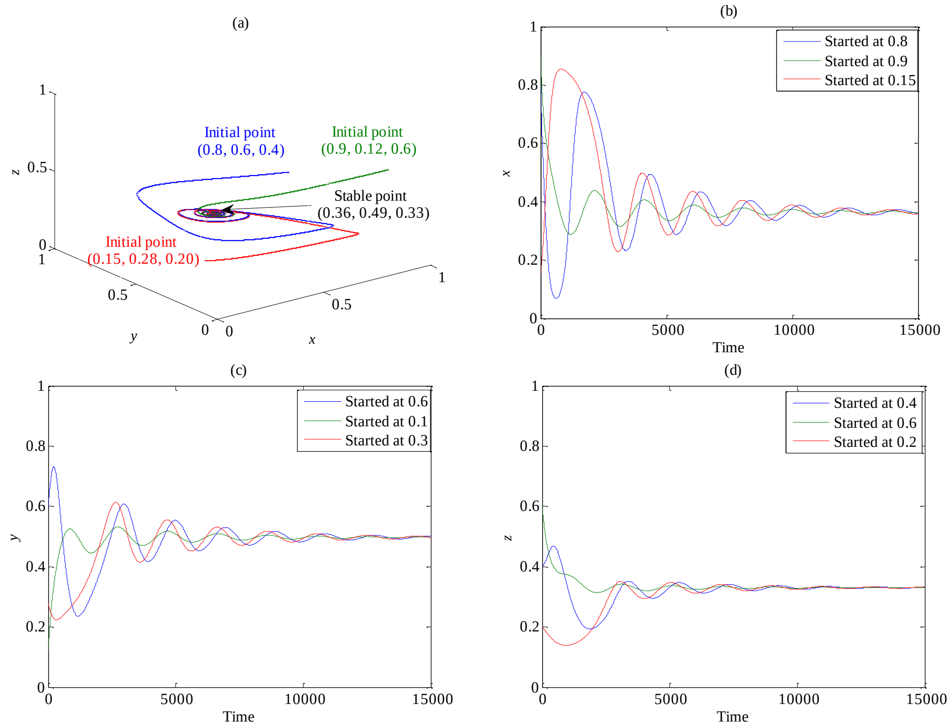

Fig. (2): (a) 3D Phase plot of system (2) for the data (24) with different initial points (b) Time series of the trajectories of x. (c) Time series of the trajectories of y. (d) Time series of the trajectories of z.

Clearly, these figures show that system (2) approaches asymptotically to the positive equilibrium point $E_2 = (0.36, 0.49, 0.33)$ starting from three different initial points and this is confirming our obtained analytical results regarding to existence of a globally asymptotically stable positive equilibrium point.

Now in order to investigate the effect of varying one parameter value on the dynamical behavior of system (2), the system is solved numerically using Range- Kutta six order method along with four steps predictor corrector method for the data given in Eq. (24) with varying one parameter value each time and the obtained results are summarized in the following table and plotted in the form of phase plots and time sires as mentioned in the following table.

Table (1): Dynamical behavior of system (2) as a function of a specific parameter with the rest of parameters as in Eq. (24).

Parameter	Range	Behavior	Figure
w_1	$w_1 < 0.55$	E_1 is asymptotically stable.	Figs. (3a) and (4a).
	$0.55 \leq w_1 < 1.15$	E_2 is asymptotically stable.	Figs. (2a)-(2d).
	$1.15 \leq w_1 < 1.80$	Periodic attractor.	Figs. (3b) and (4b).
	$w_1 \geq 1.80$	E_1 is asymptotically stable.	Figs. (3c) and (4c).
w_2	$w_2 < 0.003$	E_1 is asymptotically stable.	Similar behavior as that of w_1
	$0.003 \leq w_2 < 1.30$	E_2 is asymptotically stable.	

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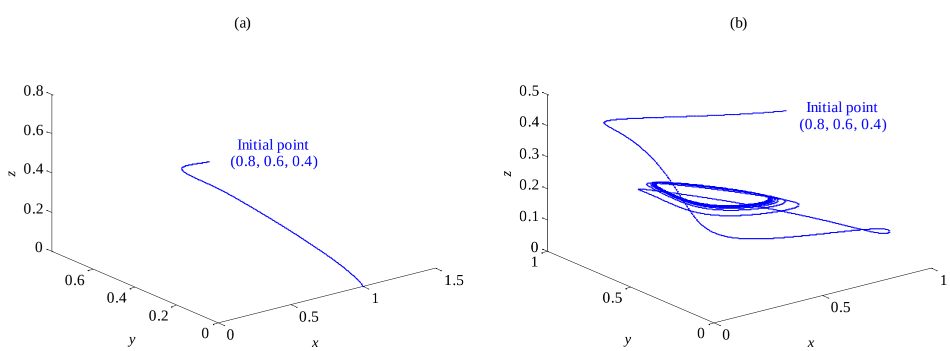
	$1.30 \leq w_2 < 1.80$	Periodic attractor.	
	$w_2 \geq 1.80$	E_1 is asymptotically stable.	
w_3	$0.04 \leq w_3 < 0.3$	E_1 is asymptotically stable.	Figs. (5a) and (6a).
	$0.3 \leq w_3 < 0.4$	Periodic attractor.	Figs. (5b) and (6b).
	$0.4 \leq w_3 < 0.75$	E_2 is asymptotically stable.	Figs. (2a) - (2d).
	$w_3 \geq 0.75$	E_1 is asymptotically stable.	Figs. (5c) and (6c).
w_4	$w_4 < 0.09$	Periodic attractor.	Figs. (7a) and (8a).
	$0.09 \leq w_4 < 0.15$	E_2 is asymptotically stable.	Figs. (2a) - (2d).
	$w_4 \geq 0.15$	E_1 is asymptotically stable.	Figs. (7b) and (8b).
w_5	$w_5 < 0.15$	Periodic attractor.	Similar behavior as that of w_4
	$0.15 \leq w_5 < 0.60$	E_2 is asymptotically stable.	
	$w_5 \geq 0.60$	E_1 is asymptotically stable.	
w_6	$w_6 < 0.080$	Periodic attractor.	Similar behavior as that of w_4
	$0.080 \leq w_6 < 0.30$	E_2 is asymptotically stable.	
	$w_6 \geq 0.30$	E_1 is asymptotically stable.	
w_7	$w_7 < 0.050$	Periodic attractor.	Figs. (9a) – (9d)
	$w_7 \geq 0.050$	E_2 is asymptotically stable.	Figs. (2a) - (2d).
e_1	$e_1 < 0.35$	E_1 is asymptotically stable.	Similar behavior as that of w_1
	$0.35 \leq e_1 < 0.55$	E_2 is asymptotically stable.	
	$0.55 \leq e_1 < 0.90$	Periodic attractor.	
	$e_1 \geq 0.90$	E_1 is asymptotically stable.	
e_2	$e_2 < 0.20$	E_1 is asymptotically stable.	Figs. (10a) and (11a).
	$0.20 \leq e_2 < 0.70$	E_2 is asymptotically stable.	Figs. (2a) - (2d).
	$e_2 \geq 0.70$	Periodic attractor.	Figs. (10b) and (11b).
n	$0 < n < 0.35$	E_2 is asymptotically stable.	Figs. (2a) - (2d).
	$0.35 \leq n < 1$	Periodic attractor.	Figs. (12) and (13).



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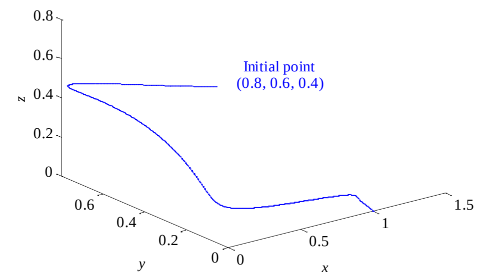

Fig. (3): 3D Phase plots of system (2) for the data (24) with different values of w_1. (a) E_1 is asymptotically stable for $w_1 = 0.4$. (b) Periodic attractor for $w_1 = 1.5$. (c) E_1 is asymptotically stable for $w_1 = 2$.

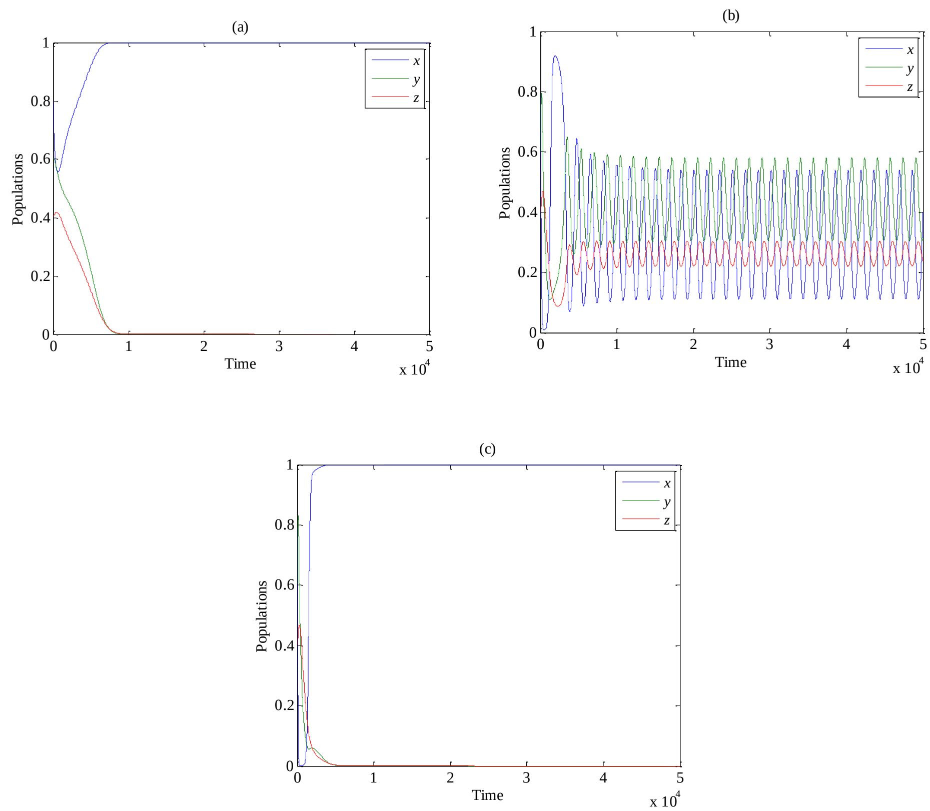

Fig. (4): (a) Time series for the attractor in Fig. (3a). (b) Time series for the attractor in Fig. (3b). (c) Time series for the attractor in Fig. (3c).

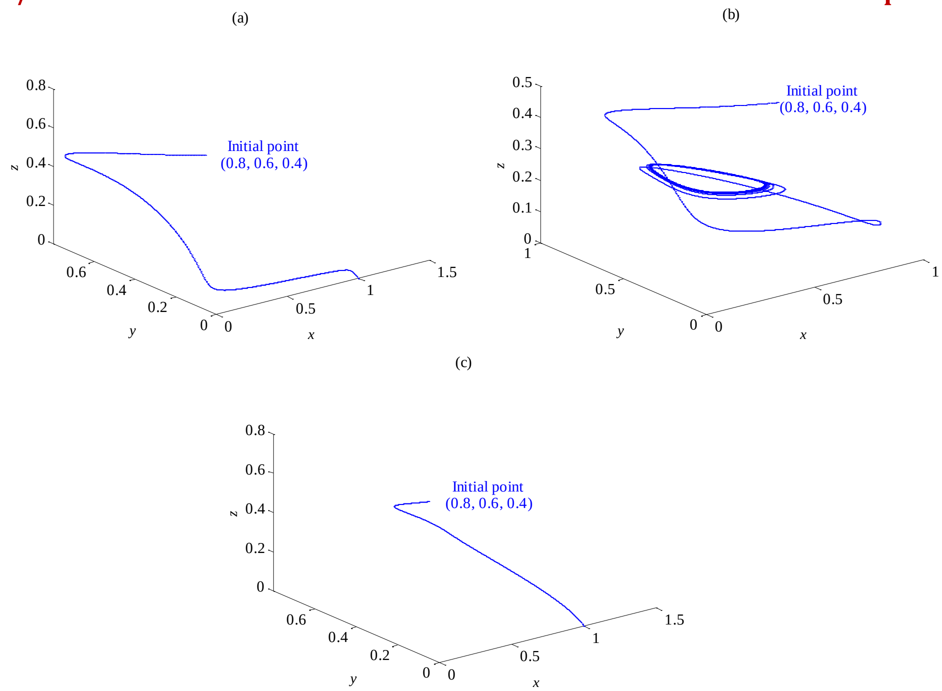

Fig. (5): 3D Phase plots of system (2) for the data (24) with different values of w_3. (a) E_1 is asymptotically stable for $w_3 = 0.20$. (b) Periodic attractor for $w_3 = 0.30$. (c) E_1 is asymptotically stable for $w_3 = 0.80$.

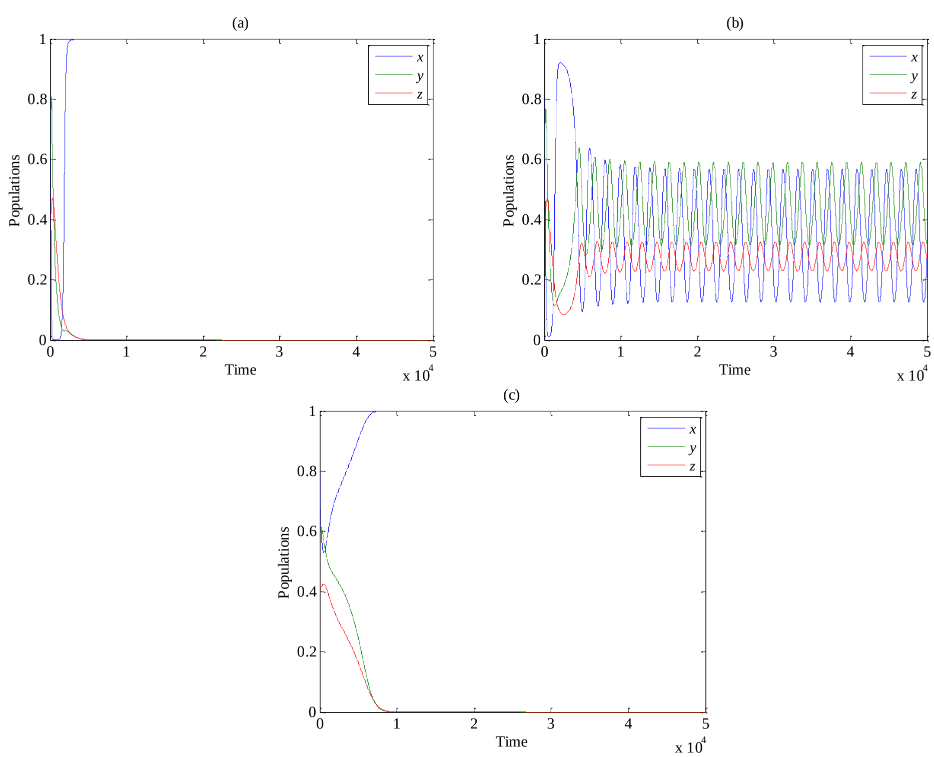

Fig. (6): (a) Time series for the attractor in Fig. (5a). (b) Time series for the attractor in Fig. (5b). (c) Time series for the attractor in Fig. (5c).

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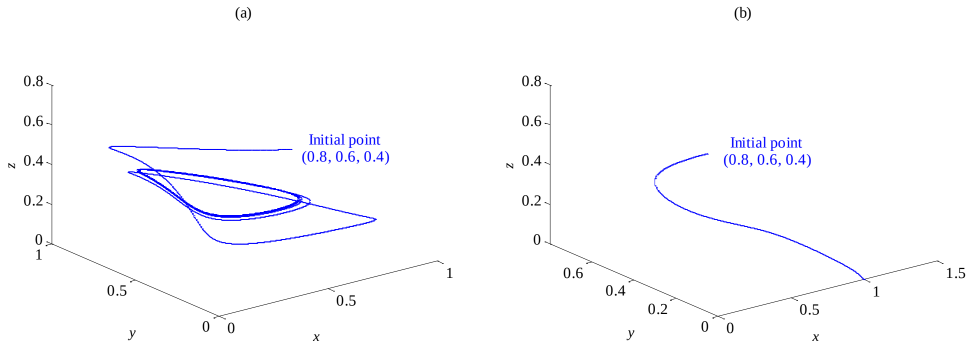


Fig. (7): 3DPhase plots of system (2) for the data (24) with different values of w_4. (a) Periodic attractor for $w_4 = 0.05$. (b) E_1 is asymptotically stable for $w_4 = 0.20$.

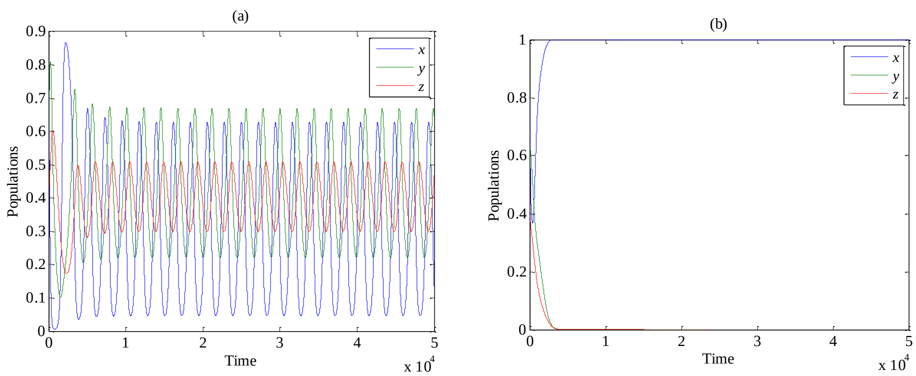
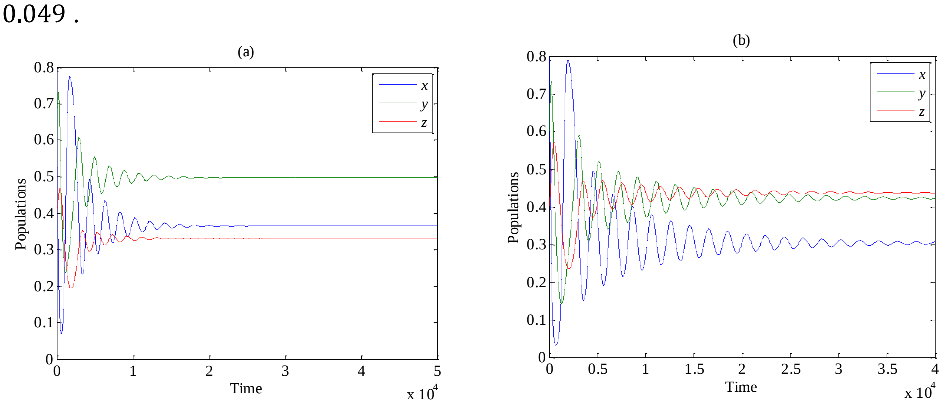


Fig. (8): (a) Time series for the attractor in Fig. (7a). (b) Time series for the attractor in Fig. (7b).

The following figure shows the occurrence of Hopf bifurcation around the positive equilibrium point as varying in the parameter w_7. Clearly, the figures given by Figs. (9a) – (9d) explain the transfers of the solution from positive asymptotically stable point to periodic dynamics for the data given by Eq. (24) as parameter w_7 passing through the value$w_7^{\sim} \cong 0.049$.



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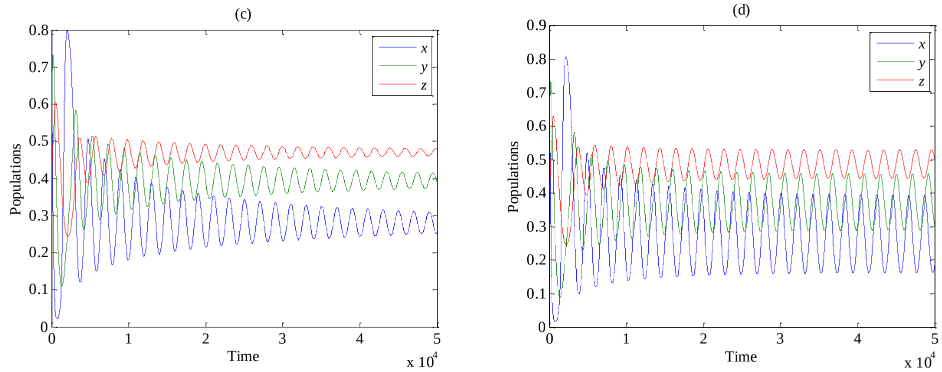

Fig.(9): Time series for the solution of system (2) for the data (24) with different values of w_7. (a) $E_2 = (0.36, 0.49, 0.33)$ is asymptotically stable point for $w_7 = 0.1$. (b) Small periodic attractor for $w_7 = 0.045$ (c) Large periodic attractor for $w_7 = 0.030$. (d) Larger periodic attractor for $w_7 = 0.020$.

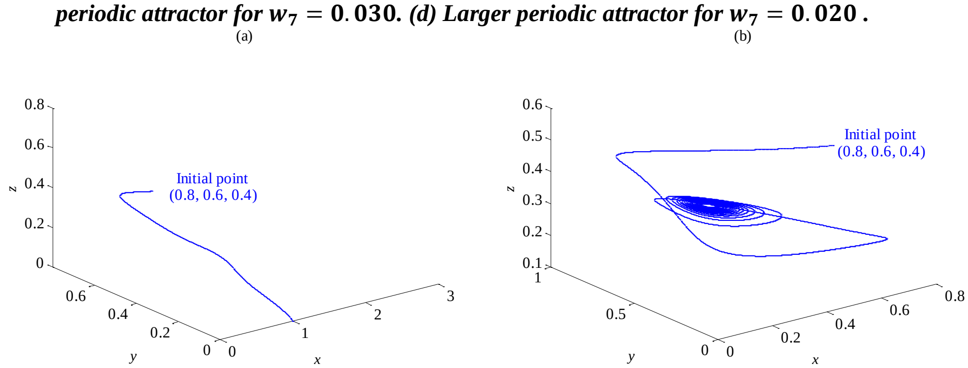

Fig. (10): 3D Phase plots of system (2) for the data (24) with different values of e_2. (a) E_1 is asymptotically stable for $e_2 = 0.11$. (b) Periodic attractor for $e_2 = 0.80$.

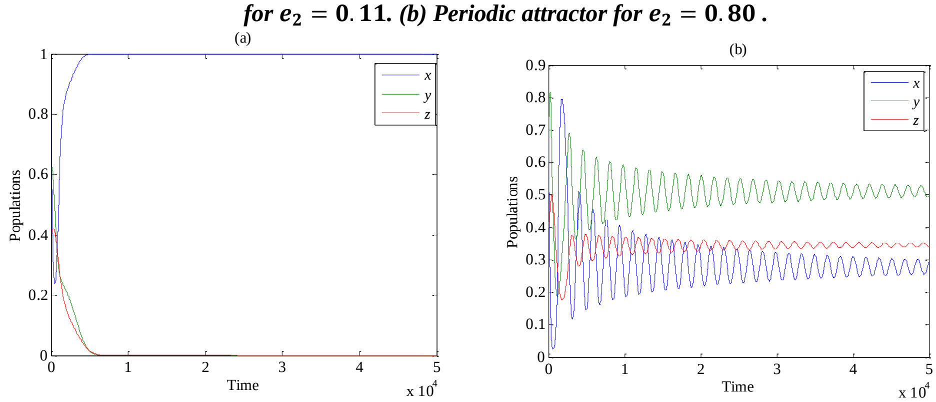

Fig. (11): (a) Time series for the attractor in Fig. (10a). (b) Time series for the attractor in Fig. (10b).

Clearly, from the below figures, it is observed that increasing the value of the vaccine rate causes decreasing in the infected predator z and then the solution of system (2) approaches to periodic dynamics near to the xy-plane.

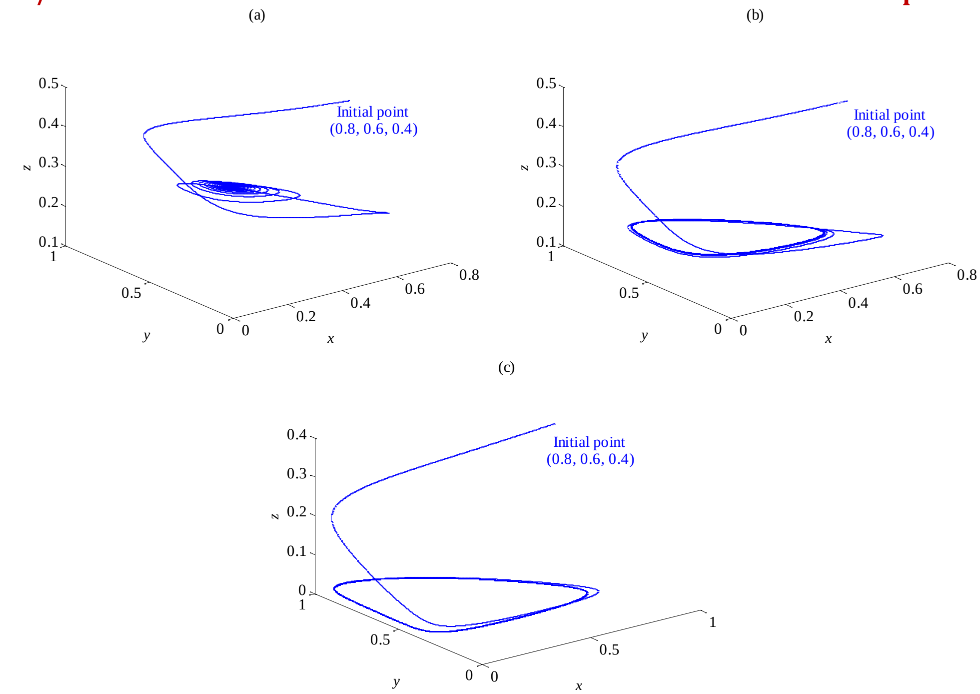

Fig. (12): 3D Phase plots of system (2) for the data (24) with different values of n. (a) Small periodic attractor for $n = 0.35$.(b) Large periodic attractor for $n = 0.60$. (c) Larger periodic attractor for $n = 0.95$.

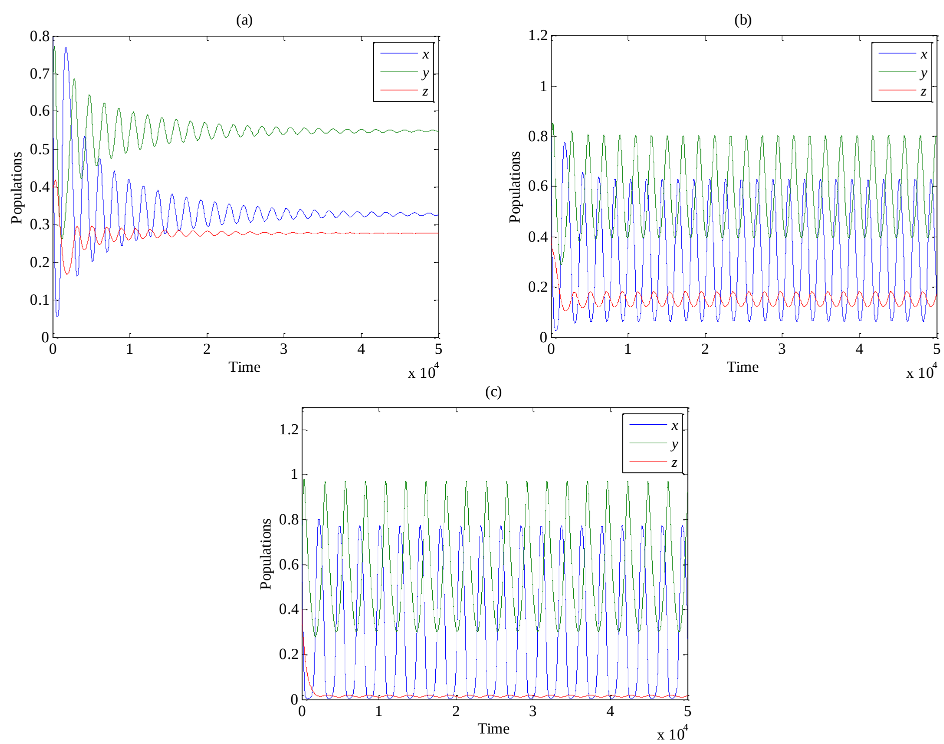

Fig. (13): (a) Time series for the attractor in Fig. (12a). (b) Time series for the attractor in Fig. (12b). (c) Time series for the attractor in Fig. (12c).

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Finally, to explain the dynamical behavior at the axial equilibrium point E_1 we used the same set of hypothetical parameters values that given in Eq. (24) and then the trajectory of system (2) is drawn in the Fig. (14) starting from three different initial points that satisfy the conditions of globally asymptotically stable sub region of E_1 (i.e. belong to the basin of attraction of E_1).

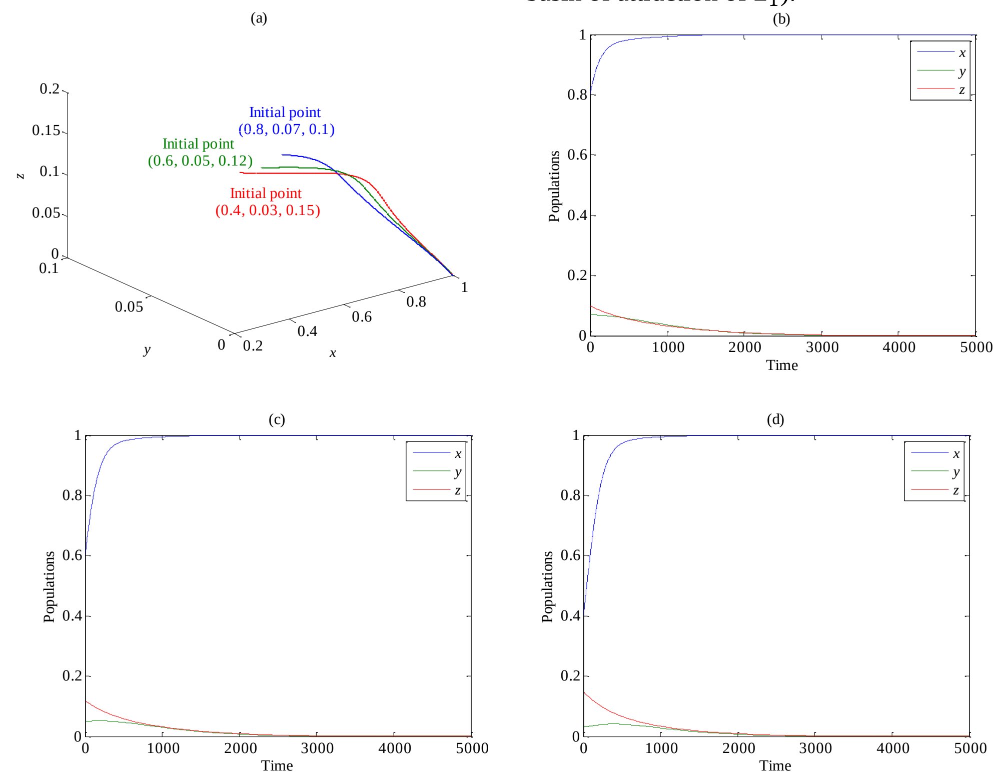

Fig. (14): (a) 3D Phase plot of system (2) for the data (24) with different initial points (b) Time series of the trajectory that starting at** $(\mathbf{0.8}, \mathbf{0.07}, \mathbf{0.1})$**. (c) Time series of the trajectory that starting at** $(\mathbf{0.6}, \mathbf{0.05}, \mathbf{0.12})$**. (d) Time series of the trajectory that starting at** $(\mathbf{0.4}, \mathbf{0.03}, \mathbf{0.15})$**.

Clearly, these figures show that the solution of system (2) approaches asymptotically to the axial equilibrium point $E_1 = (1, 0, 0)$ starting from three different initial points satisfy conditions of theorem (2) and this is confirming our obtained analytical results.

VII. CONCLUSION & DISCUSSION

In this paper, we proposed and analyzed an eco-epidemiological model that described the dynamical behavior of prey-predator model with Cosner type of functional response and linear incidence rate for describing the transfer of the disease in predator species. The model consists of three non-linear autonomous differential equations that describe the dynamics of three different populations namely prey x, susceptible predator y, infected predator z. The bounded-ness of the proposed model that given by system (2) has been discussed. The conditions for existence of the positive equilibrium point along with other possible equilibrium points are obtained. The dynamical behavior of system (2) has been investigated locally as well as globally. The local bifurcation of system (2) and the Hopf bifurcation around the positive equilibrium point are obtained. Further, it is observed that the trivial equilibrium point (E_0) always exist, and it is unstable saddle point. The axial equilibrium point (E_1) always exist, and it is always locally asymptotically stable point as well as it is globally asymptotically stable in the interior of sub region that satisfy the conditions (13a)-(13c). Finally the positive equilibrium point (E_2) of system (2) exists provided that the conditions (7)-(8) hold. It is locally asymptotically stable point provided that the conditions (12a)-(12d) hold. In addition it is a globally asymptotically stable in the interior of sub region that satisfy the conditions (14a)-(14f). On the other hand, system (2) near E_2 possesses a saddle-node bifurcation but neither transcritical nor pitchfork bifurcation can occurred provided that conditions (12a), (12b), (17a) and (17b) are hold. However, it has a Hopf bifurcation around E_2provided that the conditions (12a), (12b), (12c), (18a) and (18b) are satisfied. Finally to understand the effect of varying each parameter on the global dynamics of system (2) and to confirm our obtained analytical results, system (2) has been solved numerically and the following results are obtained for the set of hypothetical parameters values that given by Eq. (24):

1. System (2) has two types of attractors a stable point or a periodic attractor.
2. Decreasing the maximum attack rate of the susceptible predator w_1 in the range $w_1 < 0.55$ causes extinction in the predator species and the solution approaches asymptotically to axial equilibrium point E_1 indicating to occurrence of bifurcation. However increasing the value of w_1 in the range $1.15 \leq w_1 < 1.80$ leads to losing stability of the positive point and the solution approaches asymptotically to periodic dynamics in $Int. R^3_+$, indicating to occurrence of Hopf bifurcation. Finally increasing of the w_1 in the range $w_1 \geq 1.80$leads again to extinction in predator species and the system (2) approaches asymptotically to the axial equilibrium point E_1 on the x-axis too, which means to occurrence of bifurcation.
3. It is observed that varying the maximum attack rate of the infected predator w_2 or the conversion rate of the susceptible predator e_1 have similar effect as that of varying of w_1 on the dynamics of system (2).
4. Decreasing the half-saturation constant w_3 in the range $0.3 \leq w_3 < 0.4$ leads to losing stability of the positive equilibrium point and the solution approaches asymptotically to periodic dynamics in $Int. R^3_+$, indicating to occurrence of Hopf bifurcation, however further decreasing of this parameter in the range $0.04 \leq w_3 < 0.3$ causes extinction in the predator species and the solution approaches asymptotically to axial equilibrium point E_1 indicating to occurrence of bifurcation. Finally increasing of the w_3 in the range $w_3 \geq 0.75$ leads again to extinction in predator species and the system (2) approaches asymptotically to the axial equilibrium point E_1 on the x-axis too, which means to occurrence of bifurcation.
5. Decreasing the natural death rate of susceptible and infected predator w_4 in the range $w_4 < 0.09$ leads to the periodic dynamics in $Int. R^3_+$ instead of approaching to positive point, indicating to occurrence of Hopf bifurcation. Finally increasing the value of w_4 in the range $w_4 \geq 0.15$ causes extinction in the predator species and the solution approaches asymptotically to axial equilibrium point E_1 on the x-axis, indicating to occurrence of bifurcation.
6. It is observed that varying the contact infected rate w_5 or the external infected rate w_6 have similar effect as that of varying of w_4 on the dynamics of system (2).
7. Decreasing the disease death rate of the infected predator w_7 in the range $w_7 < 0.050$ leads to losing of the stability of the positive equilibrium point and the solution of system (2) approaches asymptotically to periodic dynamics, indicating to occurrence of Hopf bifurcation. Finally increasing the value of w_7 in the range $w_7 \geq 0.050$ do not have qualitative change of the dynamics of system (2) and the solution still approaches asymptotically to positive equilibrium point E_2 in $Int. R^3_+$.
8. Decreasing the conversion rate of the infected predator e_2 in the range $e_2 < 0.20$ causes extinction in the predator species and the solution approaches asymptotically to axial equilibrium point E_1 on the x-axis, indicating to occurrence of bifurcation. Finally increasing the value of e_2 in the range $e_2 \geq 0.70$ leads to losing stability of the positive point and the solution approaches asymptotically to periodic dynamics in $Int. R^3_+$, indicating to occurrence of Hopf bifurcation.
9. Decreasing the vaccine rate n in the range $0 < n < 0.35$ do not have qualitative change of the dynamics of system (2) and the solution still approaches asymptotically to positive equilibrium point E_2 in $Int. R^3_+$. However increasing the value of n in the range $0.35 \leq n < 1$ leads to instability of the positive point and the solution approaches asymptotically to periodic dynamics near to the xy-plane, indicating to occurrence of Hopf bifurcation.

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